ON THE STABILIZATION OF STATIONARY MOTIONS IN NONLINEAR CONTROL SYSTEMS

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PMM Vol.27, No.6, 1963, pp. 988-1004

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(Received July 31, 1963)

We consider the problem of stabilizing the steady-state motion of a nonlinear control system. A theory of stabilization is developed which is analogous to Liapunov stability theory according to the first approximation [1-3]. The critical cases are isolated, and the critical case of a single zero root is examined.

In the paper we use the following notation.

The matrices $\| a_{ij} \|$, $\| b_{ij} \|$, etc. are denoted by capital letters A, B, C, ... When necessary, we indicate the dimension of the matrix, for example, A_n^m, where m is the number of columns and n is the number of rows. A square matrix of degree s is denoted by $A^{(s)}$; matrices and vectors are enumerated by a subscript in parenthesis, e.g. $A_{(i)}$ and $a_{(i)}$; the letter E denotes the unit matrix; and the letter 0 denotes the null matrix. The matrix obtained from the union of A and B is denoted by

$$\|A,\ B\|,\ \left\| \begin{matrix} A \\ B \end{matrix} \right\|$$

The symbol $r(A)$ denotes the rank of A; the symbol $|A|$ denotes the determinant of the square matrix A. The spectrum of a matrix [4] is understood to be the set of its characteristic values together with their multiplicities. The notation $\lambda \in A$ means that the number λ is contained in the spectrum of A; the notation $A \subset B$ means that the columns of A are contained in B. By the symbol $\{R^n\}$ we denote an n-dimensional vector space. Lower case latin letters a, b, c, u, x, y, ... indicate vectors; the scalar product of vectors is denoted by the symbol $(a \times b) = ab$; α, β, γ, λ, ... are scalars; i, j, k, r, n, ... are indices; $|x|$ is the modulus of a vector and the absolute value of a numerical quantity; and

t is the time.

1. Formulation of the problem. We consider the control system

$$dx/dt = f(t, x, u) \qquad (x \in \{R^n\},\ u \in \{R^m\}) \tag{1.1}$$

where f is a given, sufficiently smooth vector function and x is the phase vector of the system coordinates. The vector $u(t, x)$ is the control, which we shall consider as unaffected by disturbances. The vector x is subject to small perturbations v, so that (1.1)

$$x(t) = x^*(t) + v(t) \qquad (v \in \{R^n\}) \tag{1.2}$$

where $x^*(t)$ is a given motion generated by the control $u^*(t, x^*(t))$ according to (1.1). We let

$$w = u - u^* \tag{1.3}$$

Substituting (1.2) and (1.3) into (1.1), and expanding the right-hand side in terms of the quantities v and w, we obtain an equation for the perturbed motion

$$\frac{dv}{dt} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m}\frac{\partial f}{\partial u_j}\frac{\partial u_j^*}{\partial x_i}\right)v_i + \sum_{j=1}^{m}\frac{\partial f}{\partial u_j}w_j + g(t, v, w) \tag{1.4}$$

where the derivatives are computed along the motion $x = x^*(t)$ and $u = u^*(t)$, and where $g(t, v, w)$ denotes terms which are higher than first order in v and w.

In the case $w \equiv 0$, we have the Liapunov [1] problem for the motion $v = 0$ of system (1.4). If this motion is unstable for $w \equiv 0$, then there arises the problem of the stabilization of motion (1.1), that is, the problem of choosing a control $w(t, v)$ which upon substitution into (1.4) will make the nonperturbed motion $v = 0$ asymptotically stable according to Liapunov. The function $w(t, v)$ which solves the stabilization problem, we shall call the regulator. If in addition to the requirement of asymptotic stability, one adds the condition of minimization of a certain functional of $v(t)$ and $w(t)$, then one obtains the problem of optimal stabilization or of the analytical construction of regulators [5,6].

We shall assume that w is not of lower order than v, that is

$$w(t, 0) = 0 \qquad (t \geqslant 0)$$

$$|w_j(t, v') - w_j(t, v'')| \leqslant \beta \sum_{i=1}^{n} |v_i' - v_i''| \qquad (j = 1,\ldots, m;\ \beta = \mathrm{const}) \tag{1.5}$$

for small v' and v''.

We shall also assume that the unperturbed motion $v = 0$ is stationary, that is, the right-hand side of (1.4) does not depend explicitly on t. Then system (1.4) takes on the form

$$dv/dt = Av + Bw + g(v, w) \qquad (A = \text{const}, B = \text{const}) \tag{1.6}$$

Replacing v and w anew by x and u, we consider the system

$$dx/dt = Ax + Bu + g(x, u) \tag{1.7}$$

Discarding the nonlinear terms, we obtain a system of equations for the first approximation

$$dx/dt = Ax + Bu \tag{1.8}$$

The goal of the present paper is to study the conditions under which the question of the stabilization of system (1.7) is solved by its linear approximation (1.8). Also, we separate out and classify critical cases wherein the possibility of the stabilization of (1.7) is determined by the terms $g(x, u)$, and we give the solution of the problem in the critical case of a single zero root. In this work we continue the investigations of [5-13].*

2. Preliminary remarks. We consider the matrices A_n^n and B_n^m with real or complex elements. We form the matrix

$$V = \| B, AB, \ldots, A^{(n-1)}B \| \tag{2.1}$$

Let $r(V) = r$. Then from V we may separate out r linearly independent columns which form a matrix W_n^r.

Lemma 2.1. There exists a matrix Q_r^r, and moreover it is unique, which satisfies the condition

$$AW_n^r = W_n^r Q \tag{2.2}$$

Proof. For $r = n$ the verification of the lemma is obvious, since $Q = (W^n)^{(-1)} AW^n$. We shall assume that $r < n$. Let the columns of

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* See also R.E. Kalman, J.C. Ho and K.S. Narendra, "Controllability of linear dynamical systems". Contributions to Differential Equations, *Interscience*, Vol. 1, 1962.

B_n^m be $b_{(j)}$ ($j = 1, \ldots, m$). We shall consider m square matrices $W_{(j)}^n = \| b_{(j)}, Ab_{(j)}, \ldots, A^{(n-1)}b_{(j)} \|$, each of which is of rank not higher than r. It is known [4], that the first r columns of $W_{(j)}^n$ will then be linearly independent. Making use of this circumstance for $r < n$, one may verify that the columns of the matrices $A^{(n-1} B$ and $A^{(n)}B$ can be linearly expressed in terms of the columns of the matrix $V^* = \| B, AB, \ldots, A^{(n-2)}B \|$. Furthermore, we have $AW^r \subset AV \subset \| B, AV \| = \| B, AB, \ldots, A^{(n)}B \|$, and therefore the columns of AW^r are linearly expressible in terms of the columns of V^*. But since $V^* \subset V$ and the columns of V can be expressed in terms of the columns of W^r, then the columns of V^* can also be expressed in terms of W^r, and hence the columns of the matrix AW^r as well. The latter assertion is equivalent to equation (2.2), and hence the existence of the matrix Q is proved.

We now make the contrary assumption that there exist two matrices $Q_{(1)}$ and $Q_{(2)}$, satisfying condition (2.2), that is, $AW^r = W^r Q_{(1)}$, $AW^r = W^r Q_{(2)}$.

Then $W^r(Q_{(1)} - Q_{(2)}) = 0$, and since $r(W^r) = r = \max$, then $r(Q_{(1)} - Q_{(2)}) = 0$, that is, $Q_{(1)} = Q_{(2)}$. This proves the uniqueness of the matrix Q.

To compute the matrix Q, it is necessary to solve the matrix equation (2.2), which decomposes into r systems of linear equations

$$\sum_{j=1}^{r} w_{ij} q_{jk} = p_{ik} \qquad (p_{ik} \in AW^r;\ i = 1, \ldots, n;\ k = 1, \ldots, r) \tag{2.3}$$

To the matrix W_n^r we affix the columns of a certain matrix C_n^{n-r}, the elements of which are obtained only under the condition that the matrix

$$D = \| W^r, C^{n-r} \| \tag{2.4}$$

should be nondegenerate, that is $|D| \neq 0$.

We consider a similarity transformation

$$H = D^{(-1)} AD \tag{2.5}$$

Lemma 2.2. The matrix H (2.5) has the form

$$H = \left\| \begin{matrix} A_{(1)} & A_{(2)} \\ 0 & A_{(3)} \end{matrix} \right\| \qquad (A_{(1)} = Q_r^r) \tag{2.6}$$

where Q_r^r is the matrix of equation (2.2), and the spectrum of the matrix $A_{(3)}$ is independent of the choice of C_n^{n-r} in (2.4).

Proof. Using (2.2) and (2.4), we carry out a block multiplication in (2.5)

$$H = D^{(-1)}AD = \| W^r, C^{n-r} \|^{(-1)} \| W^r Q_r^{\ r}, AC^{n-r} \| =$$
$$= \| W^r, C^{n-r} \|^{(-1)} \{ \| W^r Q_r^{\ r}, O \| + \| O, AC^{n-r} \| \} =$$
$$= \| W^r, C^{n-r} \|^{(-1)} \left\{ \| W^r, C^{n-r} \| \begin{Vmatrix} Q_r^{\ r} & O \\ O & O \end{Vmatrix} + \| O, AC^{n-r} \| \right\} = \begin{Vmatrix} Q_r^{\ r} & A_{(2)} \\ O & A_{(3)} \end{Vmatrix} \quad (2.7)$$

the result of which confirms form (2.6) of the matrix H.

We rewrite the characteristic polynomial of the matrix H (2.6)

$$| H - \lambda E | = | Q - \lambda E | \cdot | A_{(3)} - \lambda E | \quad (2.8)$$

The matrix Q in (2.2) is determined by the matrices A, B and W^r, but it is independent of C^{n-r}. Hence, the spectrum of Q is independent of C^{n-r}. Similarity transformation (2.5) does not change the spectrum of A [4], hence from (2.8) we conclude that the union of spectra of Q and $A_{(3)}$ is the spectrum of A, and further that the spectrum of $A_{(3)}$ is independent of C^{n-r}. (The matrix $A_{(3)}$ may depend on C^{n-r}.) The lemma is proved.

Note 2.1. Without loss of generality we may assume in system (1.8)

$$r(B) = \min(n, m) \quad (2.9)$$

since in the contrary case the number of equations may be reduced. Let, for example, $m < n$ and $r(B) = m - 1$. Then (changing the numbering of the columns of B if necessary) we have

$$b_{im} = \mu_1 b_{i1} + \mu_2 b_{i2} + \ldots + \mu_{m-1} b_{i,m-1}$$

Assuming

$$u_j^* = u_j + \mu_j u_m \qquad (j = 1, \ldots, m-1) \quad (2.10)$$

we obtain $B_n^{\ m} u = B_n^{\ m-1} u^*$. Hence the last column of the matrix B in (1.8) may be removed without any change in system (1.8). In this case, by (2.10) the number of equations is reduced by one.

Generally speaking, in the nonlinear system (1.7) for $r(B) < \min(n, m)$, it is impossible to reduce the number of equations in accordance with (2.10) without a change in the stabilizibility of the system. However, introducing u_j^* (2.10) and u_m instead of u_j ($j = 1, \ldots, m$) into (1.7), we can fulfill conditions (2.9) for system (1.8) without any actual change in the nonlinear system (1.7). In this circumstance, in the noncritical case of stabilization (see below, p. 1530) the linear regulator u_{j0}^* is found, after which u_m may be chosen arbitrarily. In the critical cases, the regulator u_{j0}^* stabilizes only a certain subsystem of the first approximation. The possibility of stabilizing all of system (1.7) is then determined by the terms $g(x, u)$, whereby it may turn out to be expedient to also choose the components u_m of the regulator on the basis of additional nonlinear relations (see p. 1541).

Note 2.2. If $AB = C$, then the columns of C can be linearly expressed in terms of the columns of A. Hence the columns of $A^{(k)}B$ can be expressed in terms of the columns A, and therefore

$$r(V) = r(W^r) \leqslant r(\|B, A\|) \tag{2.11}$$

and in view of (2.9) the matrix W^r for system (1.8) may be chosen in the form

$$W_n^r = \| B^m, AG^{r-m} \| \tag{2.12}$$

which will in fact be assumed below.

Note 2.3. If in system (1.8) one performs the transformation of variables $x = Dy$, where D agrees with (2.4), then in the new system $dy/dt = A_* y + B_* u$, whereby $A_* = D^{(-1)}AD = H$ in accordance with (2.5), and by virtue of (2.4) and (2.12)

$$B_* = D^{(-1)}B = \| B, AG, C \|^{(-1)} \| B, AG, C \| \begin{Vmatrix} E_m \\ O \end{Vmatrix} = \begin{Vmatrix} E_m \\ O_{n-m}^m \end{Vmatrix} \tag{2.13}$$

We form the matrices

$$V = \| B, AB, \ldots, A^{(n-1)}B \|, \quad V_* = \| B_*, HB_*, \ldots, H^{(n-1)}B_* \| \tag{2.14}$$

In view of (2.5) and (2.13) we obtain

$$V_* = D^{(-1)}V \tag{2.15}$$

Further, by virtue of (2.6), we have

$$H^{(n)} = \begin{Vmatrix} Q^{(n)} & P_{(n)} \\ O & A_{(3)}^{(n)} \end{Vmatrix} \tag{2.16}$$

where $P_{(n)}$ of the specified form is expressed in terms of the matrices Q, $A_{(2)}$ and $A_{(3)}$.

Now from (2.13), (2.14) and (2.16), it follows that

$$V_* = \left\| \begin{matrix} M_r^{n\cdot m} \\ O_{n-r} \end{matrix} \right\| \tag{2.17}$$

where

$$M = M_r^{n\cdot m} = \left\| \left\| \begin{matrix} E_m \\ O_{r-m}^m \end{matrix} \right\|, \quad Q \left\| \begin{matrix} E_m \\ O_{r-m}^m \end{matrix} \right\|, \ldots, Q^{(n-1)} \left\| \begin{matrix} E_m \\ O_{r-m}^m \end{matrix} \right\| \right\| \tag{2.18}$$

If $r(V) = r$, then by virtue of (2.15) and (2.17), we find that the rank of the matrix M (2.18) and the rank of the matrix

$$M_{*r}^{r\cdot m} = \|B_{**}, \ldots, Q^{(r-1)} B_{**}\|, \qquad B_{**} = \left\| \begin{matrix} E_m \\ O_{r-m}^m \end{matrix} \right\| \tag{2.19}$$

is also equal to r.

We introduce a definition [7] (see also the footnote on p.1523).

Definition 2.1. System (1.1) is said to be completely controlled if for arbitrary x_0, t_0 and x_1 there exists a certain time $t_1 > t_0$ and a certain control $u(t, x)$, which carries the phase (t_0, x_0) into the phase (t_1, x_1). The initial phase (t_0, x_0) is arbitrary; the final phase (t_1, x_1) includes an arbitrary vector x_1. In a stronger case, the time is also arbitrary.

The controllability of system (1.8) is tied to the rank of the matrix V. This matrix (2.1) was introduced in problems of optimality of control in [14] in connection with the condition of the generality of the state [15].

The following assertions are true [6-9].

1) System (1.8) is completely controllable if, and only if, the matrix V defined by (2.1) is of rank n. In this case the final phase (t_1, x_1) may be taken arbitrarily.

2) If the rank of the matrix V is equal to n, then there exists a linear regulator $u = Px$, such that system (1.8) becomes asymptotically stable.

3. Stabilization in terms of the first approximation. In system (1.8), according to Section 2, one may construct the matrices W^r and Q by means of the known matrices A and B and by means of (2.12) and (2.13), and thereupon find the spectra of A and Q. By virtue of Lemma 2.2, the spectrum $Q_r{}^r$ consists of certain r characteristic values of the matrix A. The following theorem is valid.

Theorem 3.1. 1) If the spectrum of the matrix Q contains all of the characteristic values of the matrix A, satisfying the conditions

$$\operatorname{Re}\lambda_{p_i} > 0 \quad (i = 1,\ldots, l); \quad \operatorname{Re}\lambda_{h_k} = 0 \quad (k = 1,\ldots, m) \quad (l+m \leqslant n)$$

then the unperturbed motion of system (1.7) is stabilized by the linear regulator $u = Px$, independent of the terms $g(x, u)$.

2) If the spectrum of the matrix Q contains not even one characteristic value λ_{p_i} of the matrix A, for which $\operatorname{Re}\lambda_{p_i} > 0$, then the unperturbed motion of system (1.7) is unstable for an arbitrary choice of the control, satisfying (1.5). Hence in this case the stabilization of systems (1.7) is impossible, independent of the terms $g(x, u)$.

3) If the spectrum of the matrix Q contains all of the characteristic values of A, for which $\operatorname{Re}\lambda_{p_i} > 0$ $(i = 1, \ldots, l)$, but contains not even one characteristic value λ_{h_k} with $\operatorname{Re}\lambda_{h_k} = 0$, then the possibility of the stabilization of system (1.7) is determined by the terms $g(x, u)$ of order higher than the first.

Proof. By means of a non-singular transformation we transform system (1.8)

$$x = Dp \tag{3.1}$$

with the matrix D defined by (2.4). We obtain

$$\frac{dp}{dt} = Hp + B_* u \quad (p \in \{R^n\}) \qquad (H = D^{(-1)}AD;\ B_* = D^{(-1)}B) \tag{3.2}$$

In view of (2.6) and (2.13), system (3.2) may be rewritten in the form

$$\frac{dy}{dt} = Qy + A_{(2)}z + \left\| \begin{matrix} E_m \\ O_{r-m} \end{matrix} \right\| u \qquad (y \in \{R^r\}) \tag{3.3}$$

$$\frac{dz}{dt} = A_{(3)}z, \qquad p = \left\| \begin{matrix} y \\ z \end{matrix} \right\| \qquad (z \in \{R^{n-r}\}) \tag{3.4}$$

Subsystem (3.4) is completely uncontrollable. We consider the system

$$\frac{dy}{dt} = Qy + \left\| \begin{matrix} E_m \\ O_{r-m} \end{matrix} \right\| u \qquad (y \in \{R^r\}) \tag{3.5}$$

A matrix of form of (2.1) for system (3.5) is equal to M_* (2.19). In view of Note 2.3, the matrix M_* has the rank r. Therefore (Section 2 (1)) system (3.5) is completely controllable.

Consider part (1) of the theorem. By assumption, the spectrum of the matrix $A_{(3)}$ satisfies the condition Re $\lambda_{r_j} < 0$ ($j = 1, \ldots, n - r$) and subsystem (3.4) is asymptotically stable. For system (3.5) we have $r(V) = r(M_*) = r$, hence by virtue of Section 2 (2) one may find a regulator

$$u = P_m^r y \qquad (y \in \{R^r\}) \tag{3.6}$$

which stabilizes system (3.5). Clearly, this regulator also stabilizes the entire system (3.2). In the variables x_i regulator (3.6) has the form

$$u = \| P_m^r, O^{n-r} \| D^{(-1)} x \qquad (x \in \{R^n\}) \tag{3.7}$$

The characteristic equation of the system

$$\frac{dx}{dt} = Ax + B \| P_m^r, O^{n-r} \| D^{(-1)} x$$

has roots only with negative real part. Thus the nonlinear system (1.7) for u (3.7) is also asymptotically stable in virtue of the theorem of Liapunov [1, p.127]. This proves Part (1) of the theorem.

Consider Part (2). By assumption, in the uncontrollable subsystem (3.4), there occurs at least one of the values λ_{p_i} with Re $\lambda_{p_i} > 0$. This subsystem is unstable, independent of the control. For an arbitrary choice $u(x)$, the system for the first approximation (1.8) will be unstable and will have a trajectory which departs from the point $x = 0$ like an exponential for $t \to \infty$. By virtue of Liapunov's theorem [1, p.128], it thus follows that system (1.7) is unstable.

Note 3.1. The theorem of Liapunov [1] was proved for analytic right-hand sides of system (1.7), but it retains its force in more general cases [2,3] and, in particular, in our case (1.5), where the right-hand sides of (1.7) satisfies the Lipschitz condition.

Consider Part (3). By assumption, in the uncontrollable subsystem (3.4) there occur only values λ_{hk} with Re $\lambda_{hk} \leqslant 0$, and among these values at least one value with Re $\lambda_{hk} = 0$. System (3.5) may be stabilized by the regulator $u = Py$. For an arbitrary specification of the control $u = Py$, the characteristic equation of system (1.8) in this case will have all roots λ_i with Re $\lambda_i \leqslant 0$ and among these roots at least one root λ_j with Re $\lambda_j = 0$. Hence in the choice of the control $u = Py$ the stability of system (1.7) is determined by the nonlinear terms $g(x, u)$ [1-3]. This then proves (3) of the theorem.

Note 3.2. For the solution of the stabilizibility problem, the determinant of the matrix Q (2.3) is not necessary. By means of the known matrices A and B, one constructs the matrix W^r, adds this to the matrix D (2.4) with the columns C^{n-r} and carries out the similarity transformation (2.5). In this circumstance, one calculates only the submatrix $A_{(3)}$ and the spectrum of its characteristic values.

The spectrum of $A_{(3)}$ complements the spectrum of Q with respect to the spectrum of the matrix A. Thus follows the criterion of stabilizibility corresponding to Parts (1) to (3) of Theorem 3.1 and based on the spectrum of $A_{(3)}$. In this case, it is essential that, by virtue of Lemma 2.2, the spectrum of $A_{(3)}$ does not depend on the choice of C^{n-r}. Without carrying out details, we remark only that by this means we arrive, in particular, at the assertions of [9], corresponding here to Parts (1) and (2) for $r(B) = 1$.

Note 3.3. For the concrete specification of a stabilizing control $u(x)$ it is not necessary to reduce system (1.8) to the form (3.3) to (3.4). If it is known that the condition for the possibility of stabilization is fulfilled, then for system (1.8) one may immediately search for a regulator $u(x)$ which solves the problem [5] for the optimal stabilization of system (1.8) in terms of the functional

$$J(u) = \int_{t_0}^{\infty} \omega(x, u)\, dt = \int_{t_0}^{\infty} \left(\sum_{i=1}^{n} a_i x_i^2 + \sum_{j=1}^{m} c_j u_j^2\right) dt = \min \tag{3.8}$$

This problem can be solved if, and only if, system (1.8) can be stabilized (see [5-9,12]).

It is now expedient to define the following ideas, the introduction and study of which comprises the basic goal of the present paper.

Definition 3.1. The case examined in Part (3) of Theorem 3.1 will be called the critical case of stabilization. Let the spectrum of the matrix A contain l characteristic roots λ_{p_i}, for which $\operatorname{Re}\lambda_{p_i} > 0$ $(i = 1, \ldots, l)$, r zero characteristic roots $\lambda_{g_k} = 0$ $(k = 1, \ldots, r)$, q pure imaginary roots $\operatorname{Re}\lambda_{h_k} = 0$, $\lambda_{h_k} \neq 0$ $(k = 1, \ldots, q)$, and s roots λ_{r_j} with $\operatorname{Re}\lambda_{r_j} < 0$ $(j = 1, \ldots, s)$. We have then $l + r + q + s = n$.

We shall say that there exists the critical case of m zero roots and p pure imaginary roots, if the spectrum of the matrix Q contains all characteristic roots λ_{p_i} with $\operatorname{Re}\lambda_{p_i} > 0$ $(i = 1, \ldots, l)$, but does not contain exactly m roots $\lambda_{g_k} = 0$ $(k = 1, \ldots, m;\ m \leqslant r)$ and exactly p imaginary roots $\operatorname{Re}\lambda_{h_k} = 0$, $\lambda_{h_k} \neq 0$ $(k = 1, \ldots, p;\ p \leqslant q)$, whereby $m \geqslant 0$, $p \geqslant 0$ and $m + p \geqslant 1$. This condition is equivalent to the condition that the spectrum of the matrix $A_{(3)}$ contain only numbers

Re $\lambda_{r_j} < 0$, m numbers $\lambda_{g_k} = 0$ ($k = 1, \ldots, m$) and p imaginary numbers Re $\lambda_{h_k} = 0$ and $\lambda_{h_k} \neq 0$ ($k = 1, \ldots, p$).

Note 3.4. If the matrices A and P of system (1.8) are real, then p is an even number. This follows from the fact that the matrix D may be chosen real, and then the matrices Q and $A_{(3)}$ will also be real.

4. The geometric criterion of stabilizibility. Let the matrix A be of simple structure, that is, let it have n linearly independent characteristic vectors. We examine system (1.8), where $r(B) = m \leqslant n$. Let the matrix V (2.1) have the rank $r (m \leqslant r \leqslant n)$. We take the matrix W^r to be of form (2.12). The columns of W^r form a subspace $\{W^r\}$ of $\{R^n\}$. We denote the characteristic vector corresponding to the characteristic number λ_j of the matrix A by the symbol $s_{(j)}$. The system of vectors $s_{(j)}$ forms a nondegenerate matrix $S = \| s_{(1)}, \ldots, s_{(n)} \|$. From the system $\{s_{(j)}\}$ we select the matrix S^{n-r}, which together with W^r forms a linearly independent system of n basis vectors

$$K = \| W^r, S^{n-r} \| = \| B^m, AG^{r-m}, S^{n-r} \| \tag{4.1}$$

The nondegenerate linear transformation

$$x = Kp \tag{4.2}$$

transforms system (1.8) into the systems

$$\frac{dp}{dt} = Hp + B_* u \tag{4.3}$$

$H = K^{(-1)}AK$ and $B_* = K^{(-1)}B$ have the form of (2.6) and (2.13), respectively.

Making use of the property $As_{(j)} = \lambda_j s_{(j)}$, we compute the matrix H. We have, as in (2.7)

$$H = K^{(-1)}AK = \| W^r, S^{n-r} \|^{(-1)} \| W^r Q^r, AS^{n-r} \| = \| W^r, s_{(i_1)}, \ldots, s_{(i_{n-r})} \|^{(-1)} \times$$

$$\times \| W^r Q^r, \lambda_{i_1} s_{(i_1)}, \ldots, \lambda_{i_{n-r}} s_{(i_{n-r})} \| = \left\| \begin{array}{c|c} Q_r^r & O \\ \hline O & \begin{matrix} \lambda_{i_1} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \lambda_{i_{n-r}} \end{matrix} \end{array} \right\| \tag{4.4}$$

where the matrix Q is computed according to (2.3) and is independent of S^{n-r}

If C^{n-r-m} is some matrix which complements $\| W^r, S^m \|$ with respect to a basis in $\{R^n\}$, then in a similar way it can be verified that transformation (4.2) with

$$K_* = \| W^r, C^{n-r-m}, S^m \| \; (m < n - r, \; S^m = \| s_{(i_1)}, \ldots, s_{(i_m)} \|)$$

transforms the matrix A into the form

$$H = K_*^{(-1)} A K_* = \left\| \begin{array}{c|c|c} Q_r & A_{(2)*}^{n-r-m} & 0 \\ \hline 0 & A_{(3)*}^{n-r-m} & \begin{matrix} \lambda_{i_1} & \ldots & 0 \\ \cdot & \cdot & \cdot \\ 0 & \ldots & \lambda_{i_m} \end{matrix} \end{array} \right\| \tag{4.5}$$

By virtue of (2.13) and (4.4) system (4.3) takes on the form

$$\frac{dy}{dt} = Qy + \left\| \begin{matrix} E_m \\ O_{r-m}^m \end{matrix} \right\| u, \qquad y \in \{R^r\} \tag{4.6}$$

$$\frac{dz_{i_k}}{dt} = \lambda_{i_k} z_{i_k}, \qquad p = \left\| \frac{y}{z} \right\| \qquad (k = 1, \ldots, n - r) \tag{4.7}$$

According to Theorem 3.1, in order to resolve the question of the possibility of stabilizing (1.8), it is necessary to determine just how many λ_{ik} enter into the matrix

$$A_{(3)} = \left\| \begin{matrix} \lambda_{i_1} & \ldots & 0 \\ \cdot & \cdot & \cdot \\ 0 & \ldots & \lambda_{i_{n-r}} \end{matrix} \right\| \tag{4.8}$$

In order to clarify the geometric picture of the distribution of the λ_j among Q and $A_{(3)}$ (4.8), we note some properties of the vectors that form the matrices W^r and S.

Property 4.1. If the numbers λ_j are different, then the matrix S contains exactly r characteristic vectors $s_{(j)}$, falling within the space $\{W^r\}$.

Indeed, the space $\{W^r\}$ cannot contain more than r characteristic vectors. If it had fewer than r, then by choosing the vector $s_{(i_k)}$ in the matrix $K_* = \| W^r, C^{n-r-1}, s_{(i_k)} \|$ one could change the spectrum of the matrix $A_{(3)}$ in the construction of (4.5), which would contradict Lemma 2.2.

By relying on the linear independence of the vectors $s_{(j)} \in S$, Property 4.1 can also be proved without the use of Lemma 2.2.

In fact, we assume that the matrix S contains only k vectors $s_{(j)}$, lying in W^r, and let these vectors form a matrix S^k, where $k < r$. Let the remaining vectors $s_{(j)} \in S$ $(j = k+1, \ldots, n)$ form a matrix S^{n-k}. The spaces $\{S^k\}$, $\{S^{n-k}\}$ and W^r have the following properties:

$$A\{S^k\} \subset \{S^k\}, \quad A\{S^{n-k}\} \subset \{S^{n-k}\}, \quad A\{W^r\} \subset \{W^r\}$$

Since $k < r$, one can find vectors $w \in \{W^r\}$, such that

$$w = \sum_{j=k+1}^{n} \mu_j s_{(j)} \quad \left(\sum \mu_{(j)}^2 \neq 0\right)$$

that is, the intersection

$$\{W_*\} = \{S^{n-k}\} \cap \{W^r\}$$

contains the non-null vectors. From the construction, it is clear that $\{W_*\}$ does not contain a single characteristic vector $s_{(j)}$ of the matrix A. However, we have $A\{W_*\} \subset \{W_*\}$. Hence it follows that $\{W_*\}$ contains at least one characteristic vector $s_{(j)}$ of the matrix A. The contradiction that we have obtained proves that $k < r$ is not possible.

Property 4.2. If there are multiple roots among the numbers λ_j, then by the simple structure of the matrix A, as is known [4], to each root λ^0 of multiplicity of p_0 there corresponds exactly p_0 linearly independent characteristic vectors $s_{(j)}{}^0$ $(j = 1, \ldots, p_0)$. Let the multiplicity of the root λ^0 in the matrix $Q_r{}^r$ be equal to k_0; $0 \leqslant k_0 \leqslant \min(r, p_0)$.

Then the linearly independent vectors $s_{(j)}{}^0$ $(j = 1, \ldots, p_0)$ may be chosen such that exactly k_0 of them fall within $\{W^r\}$.

Indeed, let p_i be the multiplicity of λ_i in the matrix A and k_i be the multiplicity of λ_i in the matrix Q, and let the vectors $s_{(ij)}$ $(j = 1, \ldots, p_i)$, corresponding to the characteristic value λ_i be chosen in some way. The matrix

$$K_{(i)}^{r+p_i} = \| W^r, s_{(i1)}, \ldots, s_{(ip_i)} \| \tag{4.9}$$

has the rank

$$r(K_{(i)}) = r + p_i - k_i \tag{4.10}$$

In fact, if

$$r(K_{(i)}) = r + m_i > r + p_i - k_i$$

then transformation (4.2) with $K_* = \| W^r, C^{n-r-m_i}, S^{m_i} \|$ would carry the matrix A into form (4.5), where the multiplicity of λ_i in $A_{(3)}$ would be equal to $m_i > p_i - k_i$. This is not possible because in the proof of Lemma 2.2 the equality $m_i = p_i - k_i$ was established. Thus

$$r\,(K_{(i)}) \leqslant r + p_i - k_i \tag{4.11}$$

for all λ_i. If the inequality in (4.11) were strict in at least one case, then (4.11) would imply that $r(\| W^r, S^n \|) < n$. Therefore the equality of equation (4.10) holds. But (4.10) means that for the numbers λ^0 one may find new characteristic vectors

$$s_{(i)}{}^{0*} = \sum_{j=1}^{p_0} \alpha_i{}^j s_{(j)}{}^0 \qquad (\alpha_i{}^j = \text{const};\ i, j = 1, \ldots, p_0) \tag{4.12}$$

such that exactly k_0 of these vectors fall within the space $\{W^r\}$.

Hence in the case of multiple roots λ_j, the simple structure of the matrix A allows one to choose characteristic vectors $s_{(i)}$ such that the matrix S will contain exactly r vectors falling within the space $\{W^r\}$.

Property 4.3. If A and B are real, the complex conjugate vectors both fall simultaneously into the space $\{W^r\}$. In fact, in this case the matrices V and W^r will be real, and hence if the vector $s_{(j)} \in \{W^r\}$, then its conjugate vector $\bar{s}_{(j)}$ belongs to $\{W^r\}$.

Using Properties 4.1, 4.2 and 4.3, we may determine $\lambda_i \in A_{(3)}$. To do this, we find the matrix S of characteristic vectors of A and determine the columns $s_{(j_i)}$, which enter into expression (4.1) for the matrix K. Namely, from the matrices

$$K_{(j)}^{r+k} = \| W^r,\ s_{(j_1)}, \ldots, s_{(j_k)} \| \qquad (k = q_j \text{ the multiplicity of } \lambda_j) \tag{4.13}$$

we isolate the submatrices of maximum rank r_j $(r \leqslant r_j \leqslant r + q_j)$, containing W^r. The columns $s_{(j_i)}$ which occur in them enter into expression (4.1) of the matrix K. The corresponding numbers λ_j, of multiplicity $r_j - r$, enter into the matrix $A_{(3)}$.

There will be exactly $n - r$ of such vectors $s_{(j_i)}$. In the case of real A and B it is sufficient to check one of the two complex conjugate characteristic vectors.

We now formulate the geometric criterion of stabilizibility.

We denote the characteristic vectors, corresponding to the characteristic numbers

$$\operatorname{Re}\lambda_{p_i} > 0 \quad (i = 1, \ldots, l), \qquad \operatorname{Re}\lambda_{h_k} = 0 \quad (j = 1, \ldots, m)$$
$$\operatorname{Re}\lambda_{r_j} < 0 \quad (k = 1, \ldots, q)$$

by the symbols

$$s_{(i)}^{+},\ s_{(k)}^{0},\ s_{(j)}^{-} \quad (l + m + q = n)$$

Theorem 4.1. 1) If all of the characteristic vectors $s_{(i)}^{+}$ ($i = 1, \ldots, l$) and $s_{(k)}^{0}$ ($k = 1, \ldots, m$) belong to the space $\{W^r\}$, then the unperturbed motion of system (1.7) is stabilized by the regulator $u = Px$, independent of the terms $g(x, u)$.

2) If at least one of the characteristic vectors $s_{(i)}^{+}$ does not belong to the space $\{W^r\}$, then the stabilization of system (1.7) is not possible.

3) If all of the characteristic vectors $s_{(i)}^{+}$ ($i = 1, \ldots, l$) belong to the space $\{W^r\}$, but at least one of the vectors $s_{(k)}^{0}$ is not contained in $\{W^r\}$, then the possibility of the stabilization of system (1.7) is determined by the term $g(x, u)$.

The validity of Theorem 4.1 follows from Theorem 3.1 in accordance with the Properties 4.1, 4.2, 4.3, Lemma 2.2 and construction (4.4) of the matrix $A_{(3)}$ (4.8) by means of transformation (4.4).

By the same means, from Theorem 4.1 there results the following assertion, which we formulate under the assumption that all of the numbers λ_j are different.

Corollary 4.1. 1) If there exist $n - r$ characteristic vectors $s_{(j)}^{-}$, not contained in the space $\{W^r\}$, then the unperturbed motion of system (1.7) is stabilized by the regulator $u = Px$, independent of $g(x, u)$.

2) If in the union of vectors $s_{(k)}^{0}$, $s_{(j)}^{-}$ there exist less than $n - r$ vectors, not contained in $\{W^r\}$, then the stabilization of system (1.7) is not possible.

3) If conditions (1) are not fulfilled, but there exist in the union of the vectors $s_{(k)}^{0}$ and $s_{(j)}^{-}$, $n - r$ vectors not contained in the space $\{W^r\}$, then the possibility of the stabilization of (1.7) depends on the terms $g(x, u)$.

Note 4.1. The case of multiple roots λ_j when the matrix A is of simple structure differs from that considered in Corollary 4.1 in that the vectors $s_{(j_k)}$ ($k = 1, \ldots, q_j$), corresponding to the root λ_j of multiplicity q_j in conditions (1) to (3), must be checked not individually, but rather the calculations in terms of the entire group, and must be carried out as many times as the vectors $s_{(j_k)}$ enter into the submatrix in (4.13) containing W^r and having maximum rank.

In accordance with Definition 3.1 we now give a geometric classification of the critical cases of stabilization.

We retain the notation $s_{(k)}^0$ ($k = 1, \ldots, m$) for the characteristic vectors corresponding to the zero roots $\lambda_{h_k} = 0$, and introduce the notation $s_{(h)}^0$ ($h = 1, \ldots, f$) for the vectors corresponding to pure imaginary roots $\operatorname{Re} \lambda_h = 0$ and $\lambda_h \neq 0$. The notations $s_{(i)}^+$ ($i = 1, \ldots, l$) and $s_{(j)}^-$ ($j = 1, \ldots, q$) remain as before; $l + m + q + f = n$.

From Theorem 4.1 it follows that the critical case of stabilization occurs when and only when all of the vectors $s_{(i)}^+$ belong to $\{W^r\}$, but at least one of the characteristic vectors $s_{(k)}^0$ and $s_{(h)}^0$ are not contained in $\{W^r\}$.

In this circumstance there occurs the critical case of k zero roots and p pure imaginary roots in all $s_{(i)}^+ \in \{W^r\}$, while in the union of m vectors $s_{(j)}^0$ ($j = 1, \ldots, m$) there exist exactly k vectors $s_{(j)}^0$ entering into the submatrix from (4.13) containing W^r and having the maximum rank $r^0 = r + k$, and while in the union f of the vectors $s_{(h)}^0$ ($h = 1, \ldots, f$) there exist exactly p vectors $s_{(h)}^0$ entering into the submatrix from (4.13) containing W^r and having the maximum rank $r^* = r + p$.

Note 4.2. **If the matrices A and B in (1.8) are real, then p is an even number.**

Now let the matrix A have complex structure. Then it is already impossible to give a geometric picture of the conditions of stabilizibility by relying only on the characteristic vectors A. In the general case we consider the matrix $T = \| T_{(1)}, \ldots, T_{(n)} \|$, transforming the matrix A to Jordan form G, that is

$$G = T^{(-1)} A \cdot T \tag{4.14}$$

Let the matrix A have l characteristic roots with $\operatorname{Re} \lambda_{p_i} > 0$, m roots with $\operatorname{Re} \lambda_{h_k} = 0$ and q roots with $\operatorname{Re} \lambda_{r_j} < 0$. We shall say that the column $T_{(s)}$ of the matrix T corresponds to the root λ_s, if λ_s stands at the s row of the main diagonal of G. The vectors $T_{(s)}$, corresponding to the roots $\operatorname{Re} \lambda_{p_i} > 0$, $\operatorname{Re} \lambda_{h_k} = 0$ and $\operatorname{Re} \lambda_{r_j} < 0$, we shall denote by the symbols $T_{(i)}^+$, $T_{(k)}^0$ and $T_{(j)}^-$, respectively.

We may now obtain a geometric criterion of stabilizibility by means of $T_{(s)}$. To do this, it is sufficient in Theorem 4.1 to replace $s_{(i)}^+$, $s_{(k)}^0$ and $s_{(j)}^-$ by $T_{(i)}^+$, $T_{(k)}^0$ and $T_{(j)}^-$.

For the proof of the validity of the criterion that has been obtained in this manner it is necessary to examine transformation (4.2), where

$$K = \| W^r, T^{n-r} \| \tag{4.15}$$

where T^{n-r} is a matrix of the columns $T_{(s_i)} \in T$ $(i = 1, \ldots, n - r)$, which together with W^r forms a basis in $\{R^n\}$. In this case the spectrum of $A_{(3)}$ in (2.6) will consist of the numbers $\lambda_{s_1}, \ldots, \lambda_{s_{n-r}}$. In fact the solution $\{y(t), z(t)\}$ of the transformed system

$$\frac{dy}{dt} = Qy + A_2 z, \quad \frac{dz}{dt} = A_{(3)} z, \qquad x = K \left\| \begin{matrix} y \\ z \end{matrix} \right\| \tag{4.16}$$

with the initial conditions $y(0) = 0$, $z_i(0) = 0$, $i \neq s_j$ and $z_{s_j}(0) = 1$ corresponds to the solutions $x(t)$ of system (1.8) with the initial conditions $x(0) = T_{s_j}$.

However, this solution $x(t)$ has the form [2, 3]

$$x(t) = P(t) e^{\lambda_{s_j} t} \tag{4.17}$$

where $P(t)$ is a vector polynomial. Therefore the solution $z(t)$ of the system

$$\frac{dz}{dt} = A_{(3)} z \tag{4.18}$$

with the initial conditions $z_i(0) = 0$ $(i \neq s_j)$, $z_{s_j}(0) = 1$ has the form

$$z(t) = Q_{s_j}(t) e^{\lambda_{s_j} t}$$

where $Q_{s_j}(t)$ is a vector polynomial since $\{y(t), z(t)\} = K^{-1} x(t)$. Hence it follows that the fundamental matrix $Z(t)$ of the solution of system (4.18) has the form

$$Z(t) = \| Q_{(s_1)}(t) e^{\lambda_{s_1} t}, \ldots, Q_{(s_{n-r})}(t) e^{\lambda_{s_{n-r}} t} \|$$

which is in accordance with the known theory for the construction of solutions of a linear system [2, 3], and proves our assertion concerning the spectrum of $A_{(3)}$.

The further proof of the criterion repeats the proof of Theorem 2.1.

Therefore the following assertion is valid.

Theorem 4.2. The critical case of the stabilizibility of system (1.7) occurs if, and only if, all of the vectors $T_{(i)}^+$ belong to $\{W^r\}$, while at least one of the vectors $T_{(k)}^0$ does not belong to $\{W^r\}$.

Note 4.3. When A has simple structure the vectors $T_{(i)}$ coincide with the characteristic vectors $s_{(i)}$ of the matrix A [4].

5. The critical case of a single zero root. Let the right-hand side of (1.8) be an analytic function in the neighborhood of the point $x = 0$. In accordance with Definition 3.1 we assume that the spectrum of the matrix Q (2.2) includes all of the characteristics roots λ_i of the matrix A with Re $\lambda_i \geqslant 0$, but does not include exactly one zero characteristic root $\lambda^0 = 0$ of it.

In accordance with Section 4 this case occurs when all of the vectors $s_{(i)}^+$ and $s_{(h)}^0$ of the simple matrix A are contained in the subspace $\{\| B, AB, \ldots, A^{(n-1)}B \|\}$, but exactly one vector $s_{(k)}^0$ $(k = 1)$ is not contained in this subspace. For multiple roots or for A of complex structure the case of a single zero root occurs only when the matrix $\| W^r, T_{(1)}^0, \ldots, T_{(m)}^0 \|$, contains all of the $T_{(k)}^0$ $(k = 1, \ldots, m)$, corresponding to the root $\lambda^0 = 0$ of multiplicity m, has rank $r + 1$, and all of the vectors $T_{(i)}^+$ and $T_{(h)}^0$ are contained in the subspace $\{\| B, AB, \ldots, A^{(n-1)}B \|\}$.

Subsystem (3.4) in this case has a single linear integral [1-3] lz = const and $l = (l_1, \ldots, l_{n-r})$ = const, to which there corresponds the linear integral

$$\xi = \| O_1{}^r L_1{}^{n-r} \| D^{(-1)} x = (0, \ldots, 0, l_1, \ldots, l_{n-r}) D^{(-1)} x = \text{const} \qquad (L_1{}^{n-r} = l) \tag{5.1}$$

of system (1.8), whatever the control $u(x)$.

Taking the quantity ξ as a new variable and assuming $x_i = v_i$ $(i = 1, \ldots, n - 1)$, we reduce the nonlinear system (1.7) to the form

$$\frac{d\xi}{dt} = X(\xi, v, u) \qquad (\xi \in \{R^1\}, u \in \{R^m\}) \tag{5.2}$$

$$\frac{dv}{dt} = A_* v + B_* u + c\xi + Y(\xi, v, u) \quad (v \in \{R^{n-1}\}) \tag{5.3}$$

Here ξ is a scalar, v is an $(n - 1)$ vector, A_* is an $(n - 1) \times (n - 1)$ matrix, B_* is an $(n - 1) \times m$ matrix, c is an $(n - 1)$ vector, and X and Y are vectors containing terms higher than first order in ξ, v and u.

It may be verified that the system

$$dv/dt = A_* v + B_* u \tag{5.4}$$

satisfies condition (1) of Theorem 3.1. Therefore, it may be stabilized by a control of type (3.6). Let the regulator found for (5.4) have the form

$$u_*(v) = P_m^{n-1}v \tag{5.5}$$

For system (5.2) to (5.3) we shall seek a regulator of the form

$$u^j(v, \xi) = u_*^j(v) + \sum_{s=1}^{\infty} \alpha_s^j \xi^s + \sum_{k=1}^{\infty} \beta_k^j |\xi|^k \qquad (j = 1, \ldots, m) \tag{5.6}$$

For $\beta_k^j = 0$ the regulator that is obtained is analytic. Generally speaking, the introduction of nonanalytic components extends the possibility of stabilization (see, for example, p.1544).

In accordance with the method of Liapunov [1-3] we consider the system

$$\varphi(\xi, |\xi|, v) = A_* v + B_* u + c\xi + Y(\xi, v, u) = 0 \tag{5.7}$$

where $u = u(\xi, |\xi|, v)$ in accordance with (5.5) and (5.6).

After the substitution of (5.6) into (5.2) to (5.3), system (5.4) takes on the form

$$dv/dt = (A_* + B_* P)\, v \tag{5.8}$$

Since $u^*(v)$ will be a regulator for (5.4), then

$$\left| \frac{\partial \varphi^i}{\partial v_j} \right|_{\xi = v = 0} = |A_* + B_* P| \neq 0 \tag{5.9}$$

Hence in a sufficiently small neighborhood of the origin $v = 0$ and $\xi = 0$ there exists a unique solution of system (5.7) which can be represented by series

$$v_i^\circ = \sum_{s+k=1}^{\infty} a_{sk}^i \xi^s |\xi|^k \qquad (i = 1, \ldots, n-1) \tag{5.10}$$

in which the coefficients a_{sk}^i of the specified form are expressed in terms of the coefficients α_s^j and β_k^j.

The function $X(\xi, v, u)$ on the right-hand side of (5.2) has the form

$$X(\xi, v, u) = \sum_{k+l+p=2}^{\infty} \sum_{i=1}^{n-1} \sum_{j=1}^{m} b_{klp}^{ij} \xi^k v_i^l u_j^p \tag{5.11}$$

Substituting (5.5), (5.6) and (5.10) into (5.11), we obtain the series

$$X^{\circ}(\xi, |\xi|) = \sum_{s+k=2}^{\infty} c_{sk}\xi^{s}|\xi|^{k} \quad (c_{sk} = c_{sk}(\alpha_s^j, \beta_k^j)) \tag{5.12}$$

Series (5.12) is the right-hand side of equation (5.2), which is obtained after a Liapunov transformation

$$v_i = w_i + v_i^{\circ} \qquad (i = 1, \ldots, n-1) \tag{5.13}$$

if one sets $w_i \equiv 0$.

Substitution (5.13) transforms system (5.2) to (5.3) to new variables ξ and w. In view of the choice of control (5.6), the transformed system may have discontinuities on the right-hand sides on the plane $\xi = 0$. However, it may be verified that this does not invalidate the arguments that have been introduced here in accordance with the scheme of Liapunov [1].

In (5.11) we introduce the notation $b_{k00}^{ij} = b_k$. If the union of terms of the lowest dimension in (5.11)

$$k + l + p = \min \tag{5.14}$$

has the form $b_k\xi^k$, then:

1) for k even or for k odd, but $b_k > 0$, the motion of system (5.2) to (5.3), and hence that of (1.7) as well, cannot be stabilized by regulator (5.6).

2) for k off, $b_k < 0$, the motion of system (1.7) is stabilized by the regulator $u_*(v)$ (5.5).

This follows immediately from the known theory of the critical case of a single zero root [1-3].

Likewise, if the terms (5.14) in (5.11) contain the variables v_i and u_j, then in the union of terms of lowest dimension

$$s + k = l \qquad (l = 2, 3, \ldots) \tag{5.15}$$

either series (5.12) will contain a function of ξ, which does not depend on the coefficients α_s^j and β_k^j (this case is similar to that considered in the previous section), or it will contain a function of ξ which depends on α_s^j or β_k^j. In the latter case we shall apply the following procedure.

We denote the sum of coefficients over all odd functions in the union of terms (5.12) satisfying (5.15) by $h_l^* = h_l^*(\alpha_s^j, \beta_k^j)$, and we shall denote the analogous sum over even functions by $h_l = h_l(\alpha_s^j, \beta_k^j)$.

Then the stabilization by control (5.6) is assured, if the coefficients α_s^j and β_k^j in (5.6) can be chosen such that the terms of lowest dimension satisfy the conditions $h_p^* < -|h_p|$, where $p = \min l \geqslant 2$.

In this circumstance it is sufficient to assume in (5.6)

$$u^j(v) = u_*^j(v) + \alpha^j\xi + \beta^j|\xi|$$

In the singular case $X(\xi, v, u) \equiv 0$ the stabilization of system (5.2) to (5.3) by regulator (5.6) is not possible, however for $u = u_*(v)$ (5.5) the unperturbed motion (5.2) to (5.3) will be stable according to Liapunov, and every perturbed motion sufficiently close to the unperturbed motion will asymptotically approach a certain stationary motion $\xi = a$ and $v_i = 0$.

If it is not possible to satisfy the condition $h_p^* < -|h_p|$, then it is at least necessary to satisfy the weakened condition $h_p^* \leqq -|h_p|$. After this it is necessary to examine in (5.12) the union of terms following in order of dimension. The remaining coefficients α_s^j and β_k^j must be arranged in such a way that this union satisfies one of the conditions

$$h_{p+1}^* < -|h_{p+1}| \quad \text{or} \quad h_{p+1}h_p < 0, \quad h_{p+1}^* < |h_{p+1}|$$

or at least one of the conditions

$$h_{p+1}h_p > 0, \quad h_{p+1}^* = -|h_{p+1}| \quad \text{or} \quad h_{p+1}h_p \leqslant 0, \quad h_{p+1}^* = |h_{p+1}| \qquad \text{etc.}$$

The regulator will be constructed if in this process there occurs a point where after a certain step $l > p$ there is fulfilled the relation

$$h_l^* < -|h_l| \tag{5.16}$$

or the relation

$$h_l h_p < 0, \quad h_l^* < |h_l| \tag{5.17}$$

wherein the quantities h_i^* and h_i satisfy, for $p \leqslant i \leqslant l - 1$, the conditions

$$\begin{aligned} &h_{i_m}h_p > 0, \qquad h_{i_m}^* = -|h_{i_m}|, \qquad p \leqslant i_m \leqslant l-1 \\ &h_{i_n}h_p \leqslant 0, \qquad h_{i_n}^* = |h_{i_n}|, \qquad p+1 \leqslant i_n \leqslant l-1 \\ &(m = 1, \ldots, r;\ n = 1, \ldots, q;\ r + q = l - p) \end{aligned} \tag{5.18}$$

If as a result of constraints, or the structure of system (5.2) to (5.3), it turns out for arbitrary possible values α_s^j and β_k^j that for certain $l \geqslant p$ and for conditions (5.18) for $p \leqslant i \leqslant l - 1$ the relation

$$h_l^* > |h_l| \tag{5.19}$$

is fulfilled or that the relation

$$h_l^* > -|h_l|, \qquad h_l h_p > 0 \tag{5.20}$$

is fulfilled, then stabilization by the method indicated does not go through.

The validity of these assertions follows from the criteria of stability for nonanalytic systems in the critical case of a single zero root [16].

An analytic regulator is obtained from (5.6) for $\beta_k^j = 0$. It is necessary to find a regulator of the form

$$u^j(v) = u_*^j(v) + \sum_{s=1}^{\infty} \alpha_s^j \xi^s \tag{5.21}$$

In this case series (5.12) will have the form

$$X^\circ(\xi) = \sum_{s=2}^{\infty} c_s \xi^s \tag{5.22}$$

We denote by c_s^* the first of the coefficients in series (5.22) which differs from zero.

Then stabilization is possible [1-3] if by a suitable choice of α_s^j one may fulfill the condition

$$c_s^* < 0 \quad (s \text{ is odd}) \tag{5.23}$$

If as a result of constraints or the structure of system (5.2) to (5.3) it is not possible to satisfy condition (5.23) and $c_s^* \neq 0$ for s even, or $c_s^* > 0$ for s odd, then system (1.7) is not stabilized by regulator (5.21).

Note 5.1). It was assumed above that the matrix B_{*n-1}^m in (5.4) satisfied condition (2.8): $r(B_*) = m = \max$. If this is not the case, then we make a change in control (2.10). Then the dimension of the regulator $u_*(v)$ (5.5) will be equal to the rank of $r(B_*)$. Since the dimension of the nonlinear regulator u (5.6) is equal to m, the vector $u_*(v)$ should be enlarged by $m - r(B_*)$ components, which may be chosen as arbitrary analytic functions of the vector v.

Let us consider an example based on an example from the book [2, p. 138].

Example. We have given the system

$$\frac{dx}{dt} = ax^2 + bxy + cy^2, \qquad \frac{dy}{dt} = y + kx + lx^2 + mxy + ny^2 + u$$

where a, b, c, k, l, m and n are constants and x, y and u are scalars. For $u \equiv 0$ the system is unstable. The linear approximation has the form

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = y + kx + u$$

We have the matrices

$$A = \begin{Vmatrix} 0 & 0 \\ k & 1 \end{Vmatrix}, \qquad B = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \qquad V = \| B, AB \| = \begin{Vmatrix} 0 & 0 \\ 1 & 1 \end{Vmatrix}, \qquad r(V) = 1 < 2$$

In the uncontrollable subsystem $dx/dt = 0$ we have the zero root $\lambda_1 = 0$. For the equation $dy/dt = y + u$ the regulator may be taken in the form $u_* = -2y$.

1) Solution in the subclass (5.21) of analytical regulators. We look for a regulator of the form (5.21)

$$u = u_* + \sum_{s=1}^{\infty} \alpha_s x^s = -2y + \sum_{s=1}^{\infty} \alpha_s x^s$$

System (5.7) has the form

$$-y + kx + lx^2 + mxy + ny^2 + \sum_{s=1}^{\infty} \alpha_s x^s = 0$$

$$y(x) = B_1 x + B_2 x^2 + B_3 x^3 + \ldots$$

where

$$B_1 = k + \alpha_1, \quad B_2 = l + mk + nk^2 + m\alpha_1 + 2nk\alpha_1 + n\alpha_1^2 + \alpha_2 = B_2^0 + \alpha_2$$

$$B_3 = (m + 2nk + 2n\alpha_1) B_2 + \alpha_3 = B_3^0 + \alpha_3$$

We have

$$X^0 = ax^2 + bxy + cy^2 = A_2 x^2 + A_3 x^3 + A_4 x^4 + \ldots$$

where

$$A_2 = a + bk + b\alpha_1 + c(k + \alpha_1)^2, \qquad A_3 = (b + 2cB_1)(B_2^0 + \alpha_2)$$

$$A_4 = cB_2^2 + (b + 2cB_1)(B_3^0 + \alpha_3)$$

The parameter α_1 is chosen by the condition $A_2 = 0$. If this is not possible (for example for $b = c = 0$ and $a \neq 0$), then stabilization is not possible. Let us assume that by the choice of α_1 we have $A_2 = 0$. If in this circumstance $b + 2cB_1 \neq 0$, then α_2 is chosen by means of the condition $A_3 < 0$; if for an arbitrary possible α_2 we have $A_3 > 0$, then stabilization is impossible. If $b + 2cB_1 = 0$, then $A_3 = 0$; and α_2 is a spare parameter which may be disposed of subsequently.

Let $b + 2cB_1 = 0$ and $A_3 = 0$. Then α_2, α_3 and α_4 are chosen by means of the condition $A_4 = 0$, $A_5 < 0$, etc. Having found the first suitable α_s ($s = 1, \ldots, s_0$), we assume the remaining α_{s_0+n} ($n = 1, 2, \ldots$) and the regulator is found.

2) Solution in the class of regulators of form (5.6). We look for a regulator of the form

$$u = u_* + \alpha x + \beta |x| = -2y + \alpha x + \beta |x|$$

We have system (5.7)

$$-y + kx + lx^2 + mxy + ny^2 + \alpha x + \beta |x| = 0$$
$$y(x, |x|) = B_1 x + B_1^* |x| + \ldots \qquad (B_1 = k + \alpha,\ B_1^* = \beta)$$

We have

$$X^\circ = ax^2 + bxy + cy^2 = A_2 x^2 + A_2^* x |x| + \ldots,$$

where

$$A_2 = a + bB_1 + cB_1^2 + cB_1^{*2} = a + b(k + \alpha) + c(k + \alpha)^2 + c\beta^2$$
$$A_2^* = bB_1^* + 2cB_1 B_1^* = [b + 2c(k + \alpha)]\beta$$

We choose α and β such that the condition $A_2^* < -|A_2|$ is fulfilled, that is

$$[b + 2c(k + \alpha)]\beta < -|a + b(k + \alpha) + c(k + \alpha)^2 + c\beta^2|$$

If it is possible to do this, then the regulator is found. For example, let $a, b, c > 0$ and $b^2 - 4ac > 0$. Then the stabilizing control will be a regulator in which $\alpha = -k$ and $\beta_2 > \beta > \beta_1$, where β_1 and β_2 are roots of the equation $c\beta^2 + b\beta + a = 0$. If for an arbitrary possible choice of α and β we have $A_2^* > -|A_2|$, then stabilization by means of regulator (5.6) is not possible.

If it is possible to find such α and β so that at least $A_2^* = -|A_2|$, then it is necessary to consider terms of the next higher dimension, choosing u in the general form (5.6).

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Translated by E.E.Z.